Technical Notes

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Improved Solutions to the Falkner-Skan Boundary-Layer Equation

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Nomenclature

c	= const
f, f', etc.	= nondimensionalized stream function and derivatives
m	= exponent in freestream velocity variation
u	= x component of velocity in boundary layer
β	= measure flow turning angle, Eq. (1)
η	= nondimensional distance measured normal to wall

Introduction

BOUNDARY layers around low-drag and supercritical airfoils are characterized by weakly accelerating and decelerating flow conditions. Such flows are amenable to methodical sensitivity analyses using the Falkner-Skan boundary-layer flow formulation of this problem. This paper summarizes the results of an extremely accurate state variable approach used to computationally solve the Falkner-Skan equation over a wide range of flow acceleration parameters β. Whereas most solutions to this problem are to three- or four-place accuracy, solutions accurate to seven places were possible using double-precision accuracy computational techniques. These extremely precise results were useful to determine the relationship between the flow acceleration parameter β and the initial velocity profile at the surface $f''(0)$ with exponential and quadratic least squares curve fits having mean square errors of less than 4×10^{-5}.

Analysis

Similarity solutions exist for the Prandtl boundary-layer equation when the velocity profile in the boundary layer varies as cx^m. Falkner and Skan[1] originally determined the governing equation for this flow problem to be

$$f''' = -ff'' + \beta(f'^2 - 1) \text{ where } \beta = 2m/(m+1) \tag{1}$$

and thought it to be applicable only in the case of incompressible two-dimensional flows. However, it was shown later by Mangler[2] that by an appropriate transformation this equation can also represent compressible flow, thereby extending the utility of solutions to the Falkner-Skan equation. Generally, the solutions to this equation are useful in both compressible and incompressible flow calculations of two-dimensional or axisymmetric problems.

The physical problems described by the Falkner-Skan equation is flow past an infinite wedge with a vertex angle of $\beta\pi$, where $0\le\beta\le2$, as illustrated in Fig. 1. Particular solutions for β are well known, with $\beta=0$ prescribing the Blasius problem for uniform flow over a flat plate and $\beta=1$ prescribing the stagnation-point flow problem.[3]

Numerous analytical studies of the Falkner-Skan problem have been performed, the most complete by Smith.[4] Other studies are described in standard textbooks.[3,5] These studies present results of varying accuracies ranging from four to eight places for certain cases. The results presented here are accurate to the seventh decimal place for f, f', f'', and f'''. Complete details are reported in Ref. 6. Results for additional values of $\beta=\pm0.01$, ±0.02, ±0.03, and ±0.07 are also available in Ref. 6, in addition to most of the values of β presented by Smith. These additional cases permit further refinement necessary to analyze weakly accelerating and decelerating flows typical in low-drag and supercritical airfoils.

Method of Solution

The full Falkner-Skan boundary-layer problem is

$$f''' = -ff'' + \beta(f'^2 - 1) \tag{2}$$

$$f = f' = 0 \text{ at } \eta = 0 \tag{3}$$

$$f' = 1 \text{ at } \eta = \infty \tag{4}$$

The method of solution used here is to replace the secondary boundary given as Eq. (4) by an initial condition on f'' at $\eta=0$, such that the condition on f' at $\eta=\infty$ is satisfied. Only one particular value of f'' at $\eta=0$ will satisfy the boundary condition at $\eta=\infty$, and this value is found by trying continually refined values of f'' until the desired limiting value is reached. The value of $f''(0)$ was found to require eight significant decimal places for $-0.1988377\le\beta\le-0.05$, nine significant decimal places for $-0.03\le\beta\le+0.040$, and ten significant decimal places for $+0.06\le\beta\le2.00$. This level of significance was determined by increasing or decreasing the last significant digit by unity to see if there was a change in $f'(\infty)$. When this condition was achieved, the last significant digit was increased to the highest value possible such that the conditions $f'(\infty)\equiv1.0000000$, $f''(\infty)\equiv+0.0000000$, and $f'''(\infty)\equiv-0.0000000$ were satisfied. For $-0.1988377\le\beta\le-0.03$ the computed results are given to $\eta=10.00$ and for $-0.02\le\beta\le2.0$ to $\eta=7.50$, which clearly shows that the infinity conditions are reached and not over- or undershot.

An example of the iterative method used has been described by Hartree[7] and Smith[4] generally as follows. Choose a value for β (say $\beta=1.0$). Assume that the limits of $f''(0)$ to be $1.23<f''(0)<1.24$. Guess the value of $f''(0)$ to be 1.235 and begin the computation until either f' exceeds unity or f''

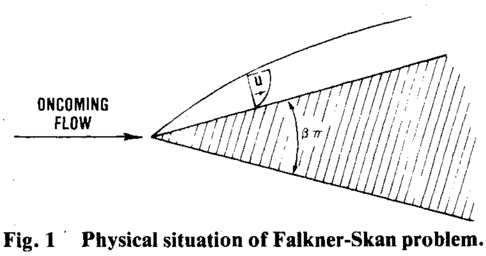

Fig. 1 Physical situation of Falkner-Skan problem.

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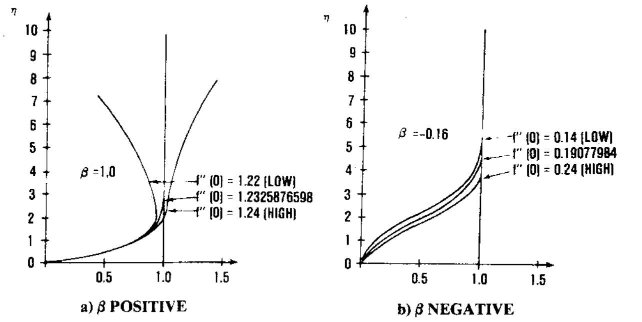

Fig. 2 Solutions to the Falkner-Skan equation.

Table 1 Effect of integration step size

Integration step size	f at $\eta = 10$	Process time, s
5×10^{-1}	0.9963940	2.6
1×10^{-1}	0.9999998	11.4
5×10^{-2}	0.9999999	22.7
1×10^{-2}	1.0000000	95.4
5×10^{-3}	1.0000000	182.8
5×10^{-3}	1.0000000	910.3

becomes negative. If f' exceeds unity before $\eta \simeq 7$, then $1.23 < f''(0) < 1.235$. Then try $f''(0) = 1.233$, and it is found again that $f'(7) < 1$, therefore $1.23 < f''(0) < 1.233$. The next guess for $f''(0)$ might be 1.232 which would give $f'(7) < 1$, therefore $1.232 < f''(0) < 1.233$. By continued application of this technique, no more than four tries are needed to improve the value of $f''(0)$ by one additional decimal place. In this particular of $\beta = 1.0$, $f''(0)$ is found to be 1.2325876598, as illustrated in Fig. 2a.

In the case of negative values of β, all values of $f''(0)$ will converge to unity for large enough values of η. However as discussed by Hartree,[7] the desired solution is the largest value of $f''(0)$ causing $f'(\infty) \rightarrow 1.0$ most rapidly from $f' < 1$ as indicated in Fig. 2b, such that $f' = 1.0000000$, $f'' = +0.0000000$, and $f''' = -0.0000000$.

Using these procedures, the Falkner-Skan problem was solved for 26 values of β. These results are reported in tabular form in Ref. 6 with their appropriate values of other boundary-layer parameters, including δ^* the displacement thickness, θ the momentum thickness, and H the shape parameter. The theoretical development of each of these parameters is described in Schlichting.[3]

Results and Discussions

As mentioned above, the detailed results of this study are tabulated in Ref. 6. The error analysis for the computational method is described elsewhere,[8,9] however an error of ± 1 in the seventh decimal place is correct throughout. The effect of integration step size was investigated for $\beta = +0.40$ with values of 5×10^{-1}, 1×10^{-1}, 5×10^{-2}, 1×10^{-2}, 5×10^{-3}, and 1×10^{-3}. Table 1 is a tabulation of the value of f' at $\eta = 10$ along with the process time required to complete the computation.

The integration step size used in this study was 5×10^{-3} which was determined by weighting the process time required for any further reduction in step size with the increased accuracy achieved. The 5×10^{-3} step size provided adequate accuracy in terms of round-off errors and convergence for the level of data significance reported. In addition, the values of δ^* and θ are accurate to ± 1 in the seventh decimal place, with H accurate to within ± 2 in the seventh decimal place. $f''(0)$ is considered accurate to ± 1 in the last significant digit.

These results may be compared with the previous studies conducted by Hartree,[7] Cope and Hartree[10] for $\beta = 0$, Rosenhead[11] for $\beta = 1$, and Smith[4] for a large range of β. Compared with Cope and Hartree at $\beta = 0$, f' and f'' differ by no more than ± 0.000004. Compared with Rosenhead at $\beta = 1$, the present calculations agree to ± 2 in the sixth decimal place. Compared with Hartree, the largest deviations in f were ± 0.0005, in f' ± 0.003, in f'' ± 0.0002, and in f''' ± 0.0002. Finally, compared with the classic study by Smith, results agree for f', f'', and f''' to within ± 0.00001.

Finally, the values of β and $f''(0)$ were analyzed using a least squares fitting routine to determine the best polynomial curve fit to the data, as well as a specified curve fit. It was found that the two best curve fits to the data were

$$\beta = -0.201211 + 0.088861 f''(0) + 0.719735 [f''(0)]^2 \quad (5)$$

having a mean square error of 4.2×10^{-5}, and

$$\beta = -0.1988377 + 0.8160218 [f''(0)]^{1.8728556} \quad (6)$$

having a mean square error of 4.7×10^{-5}. Both curve fits give $f''(0)$ to within ± 0.003 and may be useful for good first estimates.

These results demonstrate both the accuracy of the earlier studies by others (Smith notably), and at the same time provide the additional and more accurate results necessary in more refined analyses of slightly accelerated or decelerated flows.

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